

Empirical properties of the variety of a financial portfolio and the single-index model

F. Lillo and R.N. Mantegna^aMax-Planck Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany
and

Istituto Nazionale per la Fisica della Materia, Unità di Palermo, Viale delle Scienze, 90128 Palermo, Italy

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Abstract. We investigate the variety of a portfolio of stocks in normal and extreme days of market activity. We show that the variety carries information about the market activity which is not present in the single-index model and we observe that the variety time evolution is not time reversal around the crash days. We obtain the theoretical relation between the square variety and the mean return of the ensemble return distribution predicted by the single-index model. The single-index model is able to mimic the average behavior of the square variety but fails in describing quantitatively the relation between the square variety and the mean return of the ensemble distribution. The difference between empirical data and theoretical description is more pronounced for large positive values of the mean return of the ensemble distribution. Other significant deviations are also observed for extreme negative values of the mean return.

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1 Introduction

Financial markets can be regarded as model complex systems [1]. They are open systems composed of many non-equivalent sub-units interacting in a nonlinear way. They are continuously monitored and a huge amount of carefully recorded financial data are now accessible for analysis and modeling of market microstructure. This allows to perform empirical analyses elucidating statistical regularities that can be used to test models of financial activities [2–4]. These tests provide information about the strengths and weaknesses of the various models pointing out the aspects that need to be improved to obtain a better model.

Stylized facts observed in financial markets mainly refer to the statistical properties of asset returns and volatility and to the degree and nature of cross-correlation between different assets traded synchronously or quasi synchronously and belonging to given portfolios. Recently, we have proposed to model the different behavior observed in the stock returns of a portfolio by considering the statistical properties (shape, moments, etc.) of the ensemble return distribution of stocks simultaneously traded in a market. Our studies [5–7] have shown that the statistical properties of the ensemble return distribution are roughly

conserved in normal days of activity of the market whereas during crash and rally they change in a systematic way.

The single-index model [8, 9] is not adequate to model some of these findings. Specifically, it fails in describing the statistical properties of the standard deviation (called by us variety) of the ensemble return distribution and it misses to quantitatively reproduce the symmetry breaking of the empirical return distributions observed during crash and rally days [6]. On the other hand the same model is a rather attractive because it describes pretty well several stylized facts related to the first moment of the ensemble return distribution.

In the present study we investigate the empirical behavior of the variety with respect to the theoretical predictions of the single-index model. We find that variety is only mimicked at a “zero-order” by the single-index model and significant discrepancies are observed in the statistical properties of this variable both in extreme days and in periods of normal activity of the market.

2 The ensemble return distribution in normal and extreme days

In our empirical analysis we investigate the statistical properties of the ensemble return distribution obtained for a portfolio of stocks traded in a financial market. The investigated market is the New York Stock Exchange (NYSE) during the 12-year period from January 1987 to

^a *Present address:* Dipartimento di Fisica e Tecnologie Relative, Università di Palermo, Viale delle Scienze, 90128, Palermo, Italia
e-mail: mantegna@unipa.it

December 1998. This time period comprises 3032 trading days. Here we present empirical analyses of two different sets of stocks. The first is the set of all the stocks traded in the NYSE. For this statistical ensemble the number of stocks is not fixed because the total number of assets n traded in the NYSE is rapidly increasing in the investigated time period and ranges from 1128 in 1987 to 2788 in 1998. The second set is the set of 1071 stocks which are continuously traded in the NYSE in the considered period. The total number of financial records processed exceeds 6 millions.

The variable investigated in our analysis is the daily price return, which is defined as

$$R_i(t) \equiv \frac{Y_i(t) - Y_i(t-1)}{Y_i(t-1)}, \quad (1)$$

where $Y_i(t)$ is the closure price of i th asset at day t ($t = 1, 2, \dots$). In our study we consider only the trading days and we remove the weekends and the holidays from the data set. Moreover we do not consider price returns which are in absolute values greater than 50% because some of these returns might be attributed to errors in the database and may affect in a considerable way the statistical analyses. For each set of stocks, we extract the n returns of the n stocks at each trading day and we consider the probability density function (pdf) of price returns. The distribution of these returns describes the general activity of the market at the selected trading day. In the periods of normal activity of the market, the central part of the distribution is conserved for a long time. In these periods the shape of the distribution is systematically non-Gaussian and approximately symmetrical [7]. During crashes and rallies the ensemble return distribution changes abruptly shape. In a previous study [6] we have shown that the ensemble return distribution becomes asymmetric in critical days. Specifically, in crash days the ensemble return distribution becomes negatively skewed whereas in rally days the distribution becomes positively skewed. The change of the symmetry properties is not the only change of the pdf observed in crash and rally days. In fact during critical days the central moments of the pdf assume values rather different from the typical ones.

To illustrate the change of the distribution in crash and rally days and in the nearby time periods we select the three biggest crashes present during the time period of our database. They correspond to – (i) the black Monday crash of 19th October 1987 when the Standard and Poor’s 500 index had a -20.4% return, (ii) the crash of 27th October 1997 when the Standard and Poor’s 500 index had a -6.9% return, and (iii) the crash occurring at 31st August 1998 when the Standard and Poor’s 500 index had a -6.8% return. Related to these crash days there are also relevant rally days. This is because the days of greatest rallies of our database occur just one or few days after crashes. In the 1987 time period, in addition to the rally days, a second crash of -8.3% of the Standard and Poor’s 500 index occurred at 26th October 1987. The statistical behavior of stock market indices during crashes has also

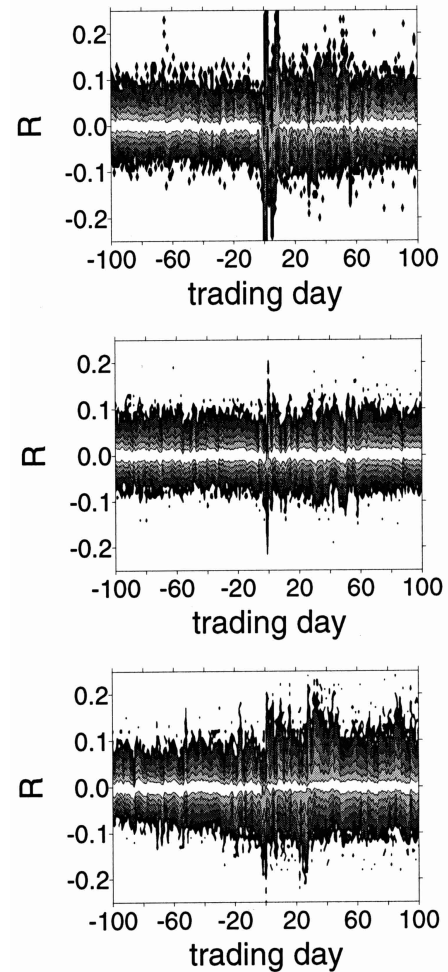


Fig. 1. Contour plots of the logarithm of the ensemble return distribution in a 200 trading days time interval centered at 19 October 1987 (top panel), 27 October 1997 (middle panel), and 31 August 1998 (bottom panel). In all the three panels we set the value 0 in the abscissa at the crash day. The contour plots are obtained for equidistant intervals of the logarithmic probability density. The brightest area of the contour plots corresponds to the most probable value.

been investigated under a different perspective in reference [10].

Figure 1 shows the contour plot of the logarithm of ensemble return distribution of the three above mentioned crash days in a 200 trading days time interval centered at day of the crisis. The contour plots of the three crises show analogies and differences. An analogy is observed by considering that the time period after the crisis is clearly characterized by a degree of high instability in the shape of the ensemble return distribution. This is shown by the behavior of the contour lines which are more parallel before crises (negative values of the trading day index in Fig. 1) than after crises. An “aftershock” period lasting more than 50 trading days is clearly detected in two of the three analyzed cases, specifically in the 1987 and in the 1998 crises. Another aspect of this analogy is that the shape of the distribution tends to fluctuate significantly

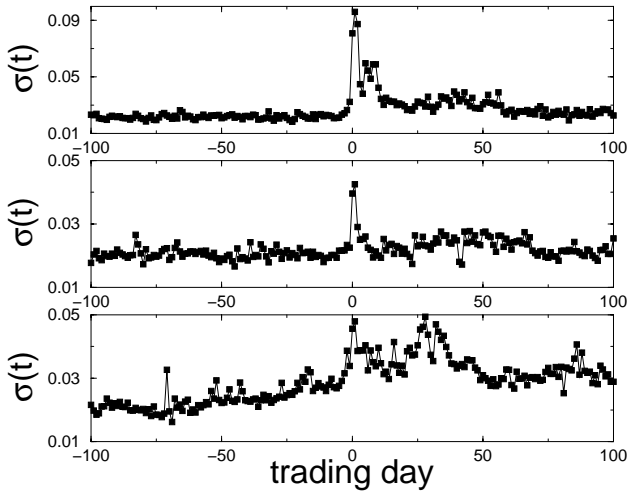


Fig. 2. Time series of the variety $\sigma(t)$ of the ensemble return distribution in a 200 trading days time interval centered at 19 October 1987 (top panel), 27 October 1997 (middle panel), and 31 August 1998 (bottom panel). In all the three panels we set the value 0 in the abscissa at the crash day. It should be noted that the scale of the y -axis is twice larger for the 1987 crisis.

during the days immediately after the crises. A difference can be noted by considering that the onset of the crises is almost abrupt for the 1987 and for the 1997 crises whereas a progressive modification of the shape of the ensemble return distribution is detected in the time period before the 1998 crisis.

In order to characterize more quantitatively the ensemble return distribution at day t , we extract the first two central moments at each of the 3032 trading days. Specifically, we consider the mean and the standard deviation of the ensemble return distribution defined as

$$\mu(t) = \frac{1}{n} \sum_{i=1}^n R_i(t), \tag{2}$$

$$\sigma(t) = \sqrt{\frac{1}{n} \left(\sum_{i=1}^n (R_i(t) - \mu(t))^2 \right)}. \tag{3}$$

The mean value of price returns $\mu(t)$ quantifies the general trend of the market at day t . The standard deviation $\sigma(t)$, *i.e.* the *variety* [5, 7] of the ensemble return distribution, measures its width. A large value of $\sigma(t)$ indicates that different companies are characterized by rather different returns at day t . In fact in days of high variety some companies perform great gains whereas others have great losses. The variety of price returns is not constant and fluctuates in time.

Figure 2 shows the variety as a function of the trading day index for the same crises shown in Figure 1. The behavior qualitatively observed in Figure 1 is now quantitatively shown in Figure 2. The abrupt onset of the 1987 and 1997 crises is rather clear, whereas a progressive increase of the variety is observed before the 1998 crisis. The presence of an aftershock period longer than 50 trading days is observed in all three cases. Wild fluctuations of the variety

are observed immediately after the 1987 and 1998 crises. In summary during a crisis and in a long period after the crisis the variety of a portfolio of stocks increases significantly. It is worth pointing out that the highest value of the variety is *not* observed at the crash day but rather at the day immediately after in all the considered cases.

The variety of a portfolio of stocks is not a variable that is invariant to time reversal. Indeed, our analysis of these case studies suggests that the behavior of the market just before and just after crises is rather different with respect to the variety of the portfolio of stocks.

3 Single-index model

We now investigate the theoretical properties of the variety of a portfolio of stocks described in terms of a single-index model. The theoretical predictions will be compared with the results of our empirical observations and with surrogate data in Section 3.2.

The single-index model [8,9] is a basic model of price dynamics in financial markets. It assumes that the returns of all stocks are controlled by one factor, usually called the “market”. In this model, for any stock i we have

$$R_i(t) = \alpha_i + \beta_i R_M(t) + \epsilon_i(t), \tag{4}$$

where $R_i(t)$ and $R_M(t)$ are the return of the stock i and of the “market” at day t respectively, α_i and β_i are two real parameters and $\epsilon_i(t)$ is a zero mean noise term characterized by a variance equal to $\sigma_{\epsilon_i}^2$. The noise terms of different stocks are assumed to be uncorrelated for $i \neq j$. Moreover the covariance between $R_M(t)$ and $\epsilon_i(t)$ is set to zero for any i . In this model each stock is correlated with the market and the presence of such a correlation induces a correlation between any pair of stocks.

There are several possible choices concerning the statistical properties of the noise terms ϵ_i . The customary choice is a Gaussian statistics [8,9] but also non-Gaussian statistics [11] has been considered. For the moment we do not specify the statistical properties of the noise terms explicitly and we only assume that the variance of each $\epsilon_i(t)$ is finite.

3.1 Central moments in the single-index model

In our study we perform both ensemble and time averages. In our notation $\langle \dots \rangle$ indicates ensemble averaged quantity, whereas $[\dots]$ indicates time averaged quantity.

The mean of the ensemble return distribution $\mu(t) \equiv \langle R_i(t) \rangle$ is given by

$$\mu(t) = \langle \alpha_i \rangle + \langle \beta_i \rangle R_M(t) + \langle \epsilon_i(t) \rangle. \tag{5}$$

The quantity $\langle \epsilon_i(t) \rangle$ is proportional to the sum of n uncorrelated random variables with zero mean and finite variance. The central limit theorem ensures that $\langle \epsilon_i(t) \rangle$ is Gaussian distributed with zero mean and variance given

by $\langle \sigma_{\epsilon_i}^2 \rangle / n$. The time average of the random variable $\mu(t)$ is given by

$$[\mu(t)] = \langle \alpha_i \rangle + \langle \beta_i \rangle [R_M(t)]. \quad (6)$$

The determination of higher moments requires the calculation of time and ensemble variances of random variables. In the following, for a random variable x_i we indicate its ensemble variance as $\langle \Delta x_i^2 \rangle \equiv \langle (x_i - \langle x_i \rangle)^2 \rangle$ whereas for a random variable $x(t)$ we indicate its time variance as $[\Delta x(t)^2] \equiv [(x(t) - [x(t)])^2]$.

The time variance of $\mu(t)$ is

$$[\Delta \mu(t)^2] = \langle \beta_i \rangle^2 [\Delta R_M(t)^2] + \frac{\langle \sigma_{\epsilon_i}^2 \rangle}{n}. \quad (7)$$

In deriving this equation we use the property that the covariance between $R_M(t)$ and $\epsilon_i(t)$ is set to zero for any i . In a similar way we compute $\langle R_i^2(t) \rangle$. By assuming that the ensemble covariance between two of α_i , β_i and ϵ_i is zero, we find that the square of the variety $\sigma(t)$ of equation (3) is equal to

$$\sigma^2(t) = \langle \Delta \alpha_i^2 \rangle + \langle \Delta \beta_i^2 \rangle R_M^2(t) + \langle \epsilon_i^2(t) \rangle - \langle \epsilon_i(t) \rangle^2. \quad (8)$$

Using equation (5) to express $R_M^2(t)$ as a function of $\mu(t)$ we rewrite equation (8) as

$$\begin{aligned} \sigma^2(t) &= \langle \Delta \alpha_i^2 \rangle + \frac{\langle \Delta \beta_i^2 \rangle}{\langle \beta_i \rangle^2} (\langle \alpha_i \rangle^2 + \langle \epsilon_i(t) \rangle^2) \\ &+ \langle \epsilon_i^2(t) \rangle - \langle \epsilon_i(t) \rangle^2 \\ &- 2 \langle \alpha_i \rangle \frac{\langle \Delta \beta_i^2 \rangle}{\langle \beta_i \rangle^2} \mu(t) + \frac{\langle \Delta \beta_i^2 \rangle}{\langle \beta_i \rangle^2} \mu^2(t). \end{aligned} \quad (9)$$

The relation between the square variety and the mean ensemble return is therefore quadratic. This implies that the single-index model predicts a linear increase of the variety for large absolute value of the mean $\mu(t)$

$$\sigma(t) \approx \frac{\sqrt{\langle \Delta \beta_i^2 \rangle}}{\langle \beta_i \rangle} |\mu(t)|. \quad (10)$$

The proportionality factor of equation (10) gives a measure of the inhomogeneity of the portfolio with respect to the market factor. Hence the increase of the variety for large values of $|\mu(t)|$ is due to the inhomogeneity of the portfolio in following the market behavior. This result is independent of the statistics of the noise terms.

Equation (10) is valid only for large values of $|\mu(t)|$. To obtain a general expression of the square variety as a function of the single-index model parameters we need to make explicit the terms $\langle \epsilon_i^2(t) \rangle$ and $\langle \epsilon_i(t) \rangle^2$ of equations (8) and (9). The term $\langle \epsilon_i^2(t) \rangle$ is proportional to the sum of the squares of n independent random variables each with mean equals to $\sigma_{\epsilon_i}^2$ and variance which is dependent on the statistical properties of ϵ_i . We indicate the time variance of $\epsilon_i^2(t)$ with v_i . By applying the central limit theorem one can show that $[\langle \epsilon_i^2(t) \rangle]$ is equals to $\langle \sigma_{\epsilon_i}^2 \rangle$ and the variance of $\langle \epsilon_i^2(t) \rangle$ is $\langle v_i \rangle / n$.

Let us now assume that the noise variables $\epsilon_i(t)$ are Gaussian. In this case the variance of ϵ_i^2 is given by $v_i = 2\sigma_{\epsilon_i}^4$. By using this property, we conclude that the first two central moments of the random variable $\langle \epsilon_i^2(t) \rangle$ are for Gaussian noise terms

$$[\langle \epsilon_i^2(t) \rangle] = \langle \sigma_{\epsilon_i}^2 \rangle \quad (11)$$

$$[\Delta \langle \epsilon_i^2(t) \rangle^2] = \frac{2}{n} \langle \sigma_{\epsilon_i}^4 \rangle. \quad (12)$$

The term $\langle \epsilon_i(t) \rangle^2$ is the square of a single Gaussian variable and is distributed according to the Gamma function $f_{a,\nu}$ [12] with $a = n/(2\langle \sigma_{\epsilon_i}^2 \rangle)$ and $\nu = 1/2$. The mean of this term is $\langle \sigma_{\epsilon_i}^2 \rangle / n$ and the variance is $2\langle \sigma_{\epsilon_i}^2 \rangle^2 / n^2$. Hence, we conclude that

$$\langle \epsilon_i^2(t) \rangle - \langle \epsilon_i(t) \rangle^2 \approx \langle \epsilon_i^2(t) \rangle. \quad (13)$$

By using equations (8), (11-12) and (13) we can explicitly write down the first two temporal moments of the random variable $\sigma^2(t)$

$$[\sigma^2(t)] = \langle \Delta \alpha_i^2 \rangle + \langle \Delta \beta_i^2 \rangle [R_M^2(t)] + \langle \sigma_{\epsilon_i}^2 \rangle \quad (14)$$

$$\begin{aligned} [\Delta(\sigma^2(t))^2] &= \\ &\langle \Delta \beta_i^2 \rangle^2 ([R_M^4(t)] - [R_M^2(t)]^2) + \frac{2}{n} \langle \sigma_{\epsilon_i}^4 \rangle. \end{aligned} \quad (15)$$

The validity of equation (14) is independent of the Gaussian assumption for the statistical properties of noise terms. On the other hand, equation (15) is valid only for Gaussian noise terms and for a market factor with finite fourth moment.

3.2 Comparison with empirical and surrogate data

In order to verify the relation between mean and variance predicted by the single-index model we generate surrogate data of an ‘‘artificial’’ market according to equation (4). The investigation of empirical data and the associated study of surrogate data are performed by considering the set of 1071 stocks traded continuously in the NYSE in the time period 1987-1998. By using the ordinary least square method we estimate the model parameters α_i , β_i and $\sigma_{\epsilon_i}^2$ for all the stocks of our ensemble. We recall that the best estimate of the model parameters does not depend on the statistical properties of noise terms. In Table 1 we show the ensemble mean and standard deviation of these parameters. The Standard and Poor’s 500 index is chosen by us as the market factor $R_M(t)$. It assumes the following values for the first two temporal moments $[R_M(t)] = 5.80 \times 10^{-4}$ and $[\Delta R_M(t)^2] = 1.02 \times 10^{-2}$ in the investigated time period. The time series of surrogate data are generated by using the above cited parameters and market factor. To check the role of the statistical properties of the noise terms we consider two different choices for $\epsilon_i(t)$ –(i) noise terms with Gaussian statistics and (ii) noise terms with non-Gaussian statistics. For the case (ii) we assume that $\epsilon_i(t) = \sigma_{\epsilon_i} w(t)$, where $w(t)$ is a random

Table 1. Value of the ensemble mean and standard deviation of the single-index model parameters obtained from empirical data with least square method by using the Standard and Poor's 500 index as market factor. The portfolio is composed by the 1071 stocks continuously traded in the New York Stock Exchange during the period 1987-1998.

Parameter	mean	standard deviation
α_i	2.02×10^{-4}	3.93×10^{-4}
β_i	6.39×10^{-1}	3.06×10^{-1}
$\sigma_{\epsilon_i}^2$	5.47×10^{-4}	6.69×10^{-4}

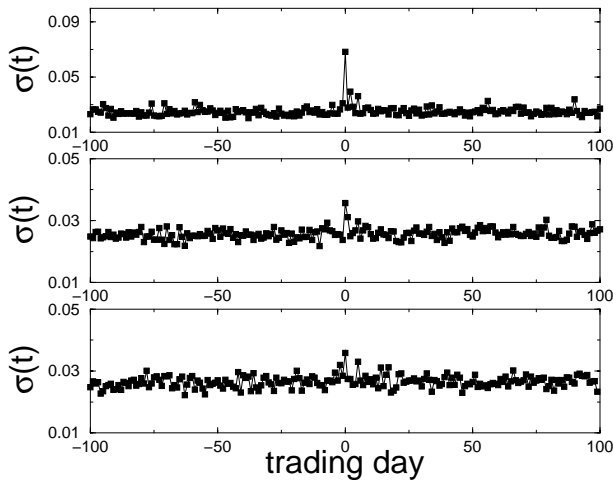


Fig. 3. Time series of the variety $\sigma(t)$ of surrogate data generated by using a single-index model with Student's t noise terms with $\kappa = 3$. The parameters of the single-index model are given in Table 1 and the market factor is the Standard and Poor's 500 index. The considered time periods are the same as in Figure 2. They consist in a 200 trading days time interval centered at 19 October 1987 (top panel), 27 October 1997 (middle panel), and 31 August 1998 (bottom panel). In all the three panels we set the value 0 in the abscissa at the crash day. The aftershock periods observed in empirical data are not present in surrogate data.

variable distributed according to a Student's t probability density function

$$P(w) = \frac{C_\kappa}{(1 + w^2/\kappa)^{(\kappa+1)/2}}, \quad (16)$$

where C_κ is a normalization constant. Empirical investigations of real data [13–15] indicates a value between 4 and 6 for the power-law exponent of $P(w)$ for large values of $|w|$. In our simulation we take the most leptokurtic distribution within this interval which corresponds to $\kappa = 3$.

We analyze the surrogate data in the same way used to investigate empirical data. In Figure 3 we show the time series of the variety corresponding to the same time periods of Figure 2 obtained for the surrogate data with Student's t noise terms. The time series of Figure 3 are rather different from the ones shown in Figure 2. A similar behavior is observed for the surrogate data with Gaussian noise terms. In Figure 3 the increases of the variety occurring

Table 2. Time average and standard deviation of $\mu(t)$ for empirical data, the theoretical prediction of the single-index model (Eqs. (6) and (7)) and surrogate data generated according to equation (4) with Gaussian and Student's t noise terms with $\kappa = 3$.

$\mu(t)$	mean	standard deviation
data	5.6×10^{-4}	73.7×10^{-4}
theory	5.7×10^{-4}	65.8×10^{-4}
Gaussian	5.8×10^{-4}	65.7×10^{-4}
Student	5.6×10^{-4}	65.9×10^{-4}

Table 3. Time average and standard deviation of $\sigma^2(t)$ for empirical data, the theoretical prediction of the single-index model (Eqs. (14) and (15)) and surrogate data generated according to equation (4) with Gaussian and Student's t noise terms with $\kappa = 3$.

$\sigma^2(t)$	mean	standard deviation
data	5.4×10^{-4}	3.8×10^{-4}
theory	5.8×10^{-4}	8.5×10^{-5}
Gaussian	5.6×10^{-4}	8.4×10^{-5}
Student	5.6×10^{-4}	2.2×10^{-4}

at the crash day is still evident in all cases but surrogate data are not able to describe the long aftershock period observed in empirical data. In other words the temporal asymmetry with respect to crash day observed in empirical data is not reproduced by the single-index model in spite of the fact that the behavior of $\mu(t)$ is well reproduced by the single-index model for all the considered time periods. This suggests that the variety of a stock portfolio is more sensitive to temporal asymmetry than the market factor $R_M(t)$. Moreover, a detailed analysis of Figure 2 shows that the day of highest variety is always the crash days in the surrogate data whereas in Section 2 we noted that the day of highest variety is the day after the crash in empirical data.

To make a quantitative comparison between empirical data, surrogate data and the theoretical predictions of the single-index model we study some statistical properties of $\mu(t)$ and $\sigma^2(t)$. Table 2 shows the temporal mean and the standard deviation of $\mu(t)$ for (i) empirical data, (ii) single-index theoretical prediction based on equations (6) and (7) and using the model parameters of Table 1, (iii) surrogate data with Gaussian statistics of the noise terms, and (iv) surrogate data with Student's t statistics of the noise terms.

The agreement between the results obtained for empirical and surrogate data with the theoretical predictions of the single-index model is pretty good. As stated in section 3.1 the mean and the standard deviation of $\mu(t)$ are not significantly affected by the statistical properties of the noise terms. This result has been observed in reference [7] in numerical simulations.

In Table 3 we show the values of the mean and standard deviation of $\sigma^2(t)$ for the same sets of data and

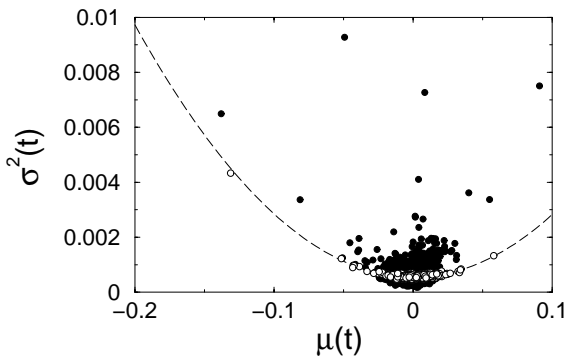


Fig. 4. Square variety $\sigma^2(t)$ of the ensemble return distribution as a function of the mean $\mu(t)$ for each trading day of the investigated time period. Each black circle refers to one trading day for empirical data. The white circles are the results obtained by analyzing surrogate data generated according to the single-index model with Gaussian noise terms. The dashed line is the theoretical prediction of equation (17) with the parameters of Table 1.

theoretical predictions as in Table 2. It is worth pointing out that the theoretical value of the standard deviation of $\sigma^2(t)$ is obtained under the assumption of Gaussian statistics for noise terms. The same quantity diverges under the assumption of Student's- t noise terms with $\kappa \leq 4$. We find that the mean value of $\sigma^2(t)$ is approximately reproduced by both Gaussian and Student's t surrogate data. The theoretical estimation of equation (14) with the model parameters of Table 1 is also close to the empirical value. However a different conclusion is obtained for the standard deviation of $\sigma^2(t)$. None of the values obtained from the theoretical estimation and numerical simulation of surrogate data is able to explain the value observed in empirical data. This is not due to the fact that the theoretical prediction is inaccurate because the theoretical prediction well describes the case of surrogate data with Gaussian noise terms. Hence, the different value detected in the empirical analysis of the standard deviation of $\sigma^2(t)$ is a clear manifestation of the limit of the single-index model. This model is able to describe the behavior of $\mu(t)$ but it is not able to describe the square variety of the portfolio properly.

To support the above conclusion we now consider in more detail the relation between $\sigma^2(t)$ and $\mu(t)$ of the ensemble return distribution. The values of the model parameters are such that equation (9) can be approximated as

$$\begin{aligned} \sigma^2(t) \approx & \langle \sigma_{\epsilon_i}^2 \rangle - 2\langle \alpha_i \rangle \frac{\langle \Delta \beta_i^2 \rangle}{\langle \beta_i \rangle^2} \mu(t) \\ & + \frac{\langle \Delta \beta_i^2 \rangle}{\langle \beta_i \rangle^2} \mu^2(t). \end{aligned} \quad (17)$$

Figure 4 shows the square of the variety $\sigma^2(t)$ as a function of the mean $\mu(t)$ for each trading day of the investigated period. In the figure each circle refers to a different trading day. The filled circles are obtained from the real data whereas the empty circles are obtained from surro-

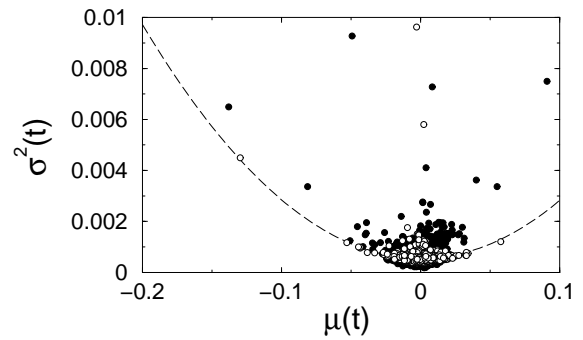


Fig. 5. Square variety $\sigma^2(t)$ of the ensemble return distribution as a function of the mean $\mu(t)$ for each trading day of the investigated time period. Each black circle refers to one trading day for empirical data. The white circles are the results obtained by analyzing surrogate data generated according to the single-index model with Student's t noise terms with $\kappa = 3$. The dashed line is the theoretical prediction of equation (17) with the parameters of Table 1.

gate data generated according to equation (4) and by using a Gaussian statistics for the noise terms. The dashed line is obtained from equation (17) by using the values of the model parameters listed in Table 1. Figure 5 is the same as Figure 4 but the empty circles are surrogate data obtained by assuming a Student's t statistics for the noise terms.

The agreement between the results obtained for the Gaussian surrogate data and the theoretical prediction is very good. This is due to the fact that the statistical uncertainty associated to the curve described by equation (17) for a set of 1071 stocks is of the order of the standard deviation of the random variable $\langle \epsilon_i^2(t) \rangle$. This quantity is the square root of equation (12) and it is equal to 3.8×10^{-5} for the model parameters of Table 1. This error is very small with respect to the scale of Figure 4 and therefore the empty circles cluster very close to the dashed line. On the other hand for the single-index model with Student's t noise terms the standard deviation of $\langle \epsilon_i^2(t) \rangle$ diverges and for this reason the dashed curve in Figure 5 describes well the average behavior of the square variety but large fluctuations around the average behavior are observed.

We finally compare the theoretical prediction and the behavior of surrogate data with the results obtained in the empirical analysis (filled circles in Figs. 4 and 5). The empirical data are on average approximately described by the single-index model but the fluctuation of the empirical results around the theoretical line is much larger than the one predicted by the single-index model with Gaussian noise terms and it is larger than the one observed in the simulated surrogate data with Student's t noise terms. Moreover, in the presence of large values of $|\mu(t)|$ the discrepancy between the empirical data and the prediction of the single-index model becomes progressively more pronounced. In particular, for large values of $|\mu(t)|$ the square variety is much larger than the value predicted by the single-index model. For large values of $|\mu(t)|$, we observe a different behavior in crash and rally days. Specifically,

the variety of the portfolio has a relative increase with respect to the theoretical prediction of the single-index model which is larger for rally than for crash days.

4 Conclusions

The present study investigates the behavior of the variety of a portfolio of stocks in normal and extreme days of market activity. The variety of a portfolio of stocks carries information about the market activity which is not included in simple models of financial markets such as the single-index model. The time evolution of the variety shows a breaking of temporal symmetry at the crash days. In fact aftershock periods of the variety are clearly observed only in empirical data whereas the surrogate data of the single-index model show a time evolution which is approximately time reversal. This is observed even if one uses the empirical time series of the Standard and Poor's 500 index as market factor. In other words the time series of the variety is showing the time arrow much better than any market factor. Independent evidence of absence of time reversal of the statistical properties of financial time series has been given in reference [16]

A second point considered in our study concerns the value of the variety observed in empirical data and the difference between it and the value predicted by the single-index model. The single-index model is able to mimic the average behavior of the square variety but fails in describing quantitatively the correct relation between the square variety and the mean return of the ensemble distribution. In particular the difference between empirical data and theoretical description is more pronounced for large positive values of the mean return of the ensemble distribution. Other significant deviations are also observed for extreme negative values of the mean return. A large spreading around the theoretical curve is observed in the entire $\mu(t)$ axis. This spreading cannot be simply explained as statistical uncertainty due to the presence of noise terms. In fact surrogate data both with Gaussian and Student's t noise terms are able to explain only part of the spreading.

A possible interpretation of the deviations of the empirical values of the square variety from the theoretical predictions of the single-index model observed in crash and rally days of financial markets is that the portfolio of stocks undergoes a change of the β_i parameters in the presence of large movements of the market. Under this interpretation, the different behavior in crash and rally days could reflect the existence of a different degree

of homogeneity of the market. In particular, within this framework, empirical data indicates that the market inhomogeneity increases more during rally than during crash days.

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